Monthly Notices
of the
ROYAL ASTRONOMICAL SOCIETY

MNRAS **483,** L85–L87 (2019)
Advance Access publication 2018 November 23
doi:10.1093/mnrasl/sly222

Gravitational potential and non-relativistic Lagrangian in modified gravity with varying G

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Accepted 2018 November 16. Received 2018 November 12; in original form 2018 October 28

ABSTRACT

We have recently shown that the baryonic Tully–Fisher (BTF) and Faber–Jackson (BFJ) relations imply that the gravitational 'constant' G in the force law vary with acceleration a as $1/a$. Here we derive the converse from first principles. First we obtain the gravitational potential for all accelerations and we formulate the Lagrangian for the central-force problem. Then action minimization implies the BTF/BFJ relations in the deep MOND limit as well as weak-field Weyl gravity in the Newtonian limit. The results show how we can properly formulate a non-relativistic conformal theory of modified dynamics that reduces to MOND in its low acceleration limit and to Weyl gravity in the opposite limit. An unavoidable conclusion is that a_0, the transitional acceleration in modified dynamics, does not have a cosmological origin and it may not even be constant among galaxies and galaxy clusters.

Key words: gravitation – methods: analytical – galaxies: kinematics and dynamics.

1 INTRODUCTION

In previous work (Christodoulou & Kazanas 2018), we showed that in the regime in which the observed baryonic Tully–Fisher (BTF; Tully & Fisher 1977; McGaugh et al. 2000; McGaugh 2012) and Faber–Jackson (BFJ; Faber & Jackson 1976; Sanders 2009; den Heijer et al. 2015) relations are valid, the gravitational 'constant' G should vary with acceleration a in the force law. Such a varying $G(a)$ function can naturally account for the non-Newtonian force postulated in Modified Newtonian Dynamics (MOND; Milgrom 1983a,b,c, 2015a,b, 2016), as well as for additional terms that appear only in weak-field Weyl gravity (Mannheim & Kazanas 1989, 1994).

In the deep MOND limit of $a \ll a_0$, where a_0 is a transitional acceleration and $G = G_0 a_0/a$, the variation of G introduces only one universal constant, the product $G_0 a_0$ (see also Milgrom 2015c). Furthermore, the Weak Equivalence Principle remains valid since the inertial mass is not modified, whereas the Strong Equivalence Principle is invalid since G varies at all scales. These findings suggest that a_0 may not have a cosmological origin despite the well-known numerical coincidence that $a_0 \simeq cH_0 \simeq 1.2 \times 10^{-10}$ m s^{-2}, where c is the speed of light and H_0 is the Hubble constant. In fact, G_0 and a_0 may not even be constants among galaxies or clusters of galaxies; they may vary in space in a way that maintains their universal constant product.

This last statement may not be entirely clear: We measure G_0 in the laboratory at high accelerations and the measured value works well at Solar system scales. But our value of G_0 for $a \gg a_0$ is not independently constrained by a relation such as the BTF/BFJ relations for $a \ll a_0$. So we have no independent evidence that G_0 takes the same value at the centre of our Galaxy or in other galaxies for that matter. This has become a major point of contention recently and we will return to it in Section 4.

Our previous work relied on important galaxy observations (Faber & Jackson 1976; Tully & Fisher 1977) to establish a theoretical result, namely, that $G \propto 1/a$ at very low accelerations. In this work, we demonstrate that the converse is also true and that it effectively ties up the existence of the BTF/BFJ relations to a single fundamental assumption, the variation of $G(a)$ in the force law. We formulate our derivations by obtaining the gravitational potential and the associated non-relativistic Lagrangian of the central-force problem with varying $G(a)$ and then by considering the radial Euler–Lagrange equation in spherical symmetry.

In Section 2, we derive the gravitational potential and the Lagrangian in the general case that includes the asymptotic cases as well as the intermediate accelerations regime. In Section 3, we derive the BTF/BFJ relations and their first-order corrections as a special case in which $a \ll a_0$; as well as weak-field Weyl gravity as a correction term to Newtonian gravity in the Newtonian limit

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$a \gg a_0$. In Section 4, we discuss our results in light of the latest developments in the field.

2 GRAVITATIONAL POTENTIAL AND LAGRANGIAN IN THE CENTRAL-FORCE PROBLEM WITH VARYING G

In the general case, applicable to all accelerations irrespective of magnitude a, the function $G(a)$ is given by the equation

$$G(a) = G_0 + \frac{G_0 a_0}{a}\,, \tag{1}$$

where $G_0 = 6.674 \times 10^{-11}$ m^3 kg^{-1} s^{-2} is the Newtonian value of the gravitational constant and $G_0 a_0 = 8.0 \times 10^{-21}$ m^4 kg^{-1} s^{-4} is a new characteristic constant that appears in the deep MOND limit (Milgrom 2015c). This new constant has dimensions of $[v]^4/[M]$, a strong hint that it is related to the BTF/BFJ relations (see Section 3 below).

The two terms in equation (1) are mandatory in order for $G(a)$ to describe correctly the gravitational force in the above two asymptotic cases. These terms combine to also describe the regime of intermediate accelerations. One might be tempted to use different functional forms for $G(a)$ with the appropriate limiting behaviours, as was also done for MOND with its arbitrary interpolating functions, but such different forms introduce additional spurious physics in the intermediate regime. For this reason, adoption of equation (1) affords us less freedom in modifying the force law as compared to MOND whose dynamics depends only on the asymptotic form of the force and treats the intermediate regime as free of additional constraints.

2.1 Gravitational potential

When the force law $a = G(a)M/r^2$ is modified by the varying $G(a)$, the gravitational potential $\Phi(r)$ of a central mass M at distance r is no longer equal to its Newtonian form $G(a)M/r$. Here we derive $\Phi(r)$ from the acceleration a by integrating the equation

$$a \equiv -\frac{\mathrm{d}}{\mathrm{d}r}\Phi(r)\,, \tag{2}$$

where a is derived from force balance (Christodoulou & Kazanas 2018), viz.[1]

$$a = \frac{a_\mathrm{N}}{2}\left(1 + \sqrt{1 + \frac{4a_0}{a_\mathrm{N}}}\right)\,, \tag{3}$$

where the Newtonian acceleration a_N is defined by

$$a_\mathrm{N} \equiv \frac{G_0 M}{r^2}\,. \tag{4}$$

In equation (2), $\Phi(r)$ is defined implicitly without the customary negative sign so that the magnitudes of the accelerations will be strictly positive, viz. $a > 0$ and $a_\mathrm{N} > 0$.

Substituting equations (3) and (4) into equation (2) and carrying out the integration over r, we find that

$$\frac{\Phi(x)}{\sqrt{G_0 M a_0}} = \frac{1 + \sqrt{1 + 4x^2}}{2x} - \ln\left(2x + \sqrt{1 + 4x^2}\right)\,, \tag{5}$$

[1]Equation (3) happens to be one of MOND's interpolating functions (the 'simple' function; e.g. Famey & McGaugh 2012) and agrees very well with the empirical results of McGaugh, Lelli & Schombert (2016) and Lelli et al. (2017) who measured the acceleration at ∼3000 distinct points in 153 and 240 galaxies, respectively.

where the dimensionless radius x is defined by

$$x \equiv r/r_M\,, \tag{6}$$

and the MOND characteristic radius r_M is given by

$$r_M = \sqrt{G_0 M/a_0}\,. \tag{7}$$

In the Newtonian limit $x \to 0$, equation (5) reduces to $\Phi(r) \approx G_0 M/r - a_0 r$ and the acceleration (equation 2) then is $a \approx a_\mathrm{N} + a_0$. The Newtonian term a_N was expected, whereas the non-Newtonian constant term has only been predicted in the weak-field limit of conformal Weyl gravity (Mannheim & Kazanas 1989, 1994).

In the deep MOND limit $x \to \infty$, equation (5) reduces to $\Phi(r) \approx -\sqrt{G_0 M a_0}\ln r + G_0 M/(2r)$ and the acceleration (equation 2) then is $a \approx \sqrt{a_\mathrm{N} a_0} + a_\mathrm{N}/2$ (see also Christodoulou & Kazanas 2018).

2.2 Lagrangian formulation

The Lagrangian of a test particle orbiting around mass M at distance r is written in polar coordinates (r, θ) as

$$\mathcal{L}(r, v) = \frac{1}{2}v^2 - \Phi(r)\,, \tag{8}$$

where the orbital speed $v = r(\mathrm{d}\theta/\mathrm{d}t)$, t is the time, and the potential Φ (equation 5) is written again without the negative sign to ensure that $a > 0$. An alternative form can be produced by using the constant specific angular momentum of the test particle $\ell = r\,v$ to eliminate v, but the calculations are actually easier when using equation (8).

The radial Euler–Lagrange equation for $\mathcal{L}(r, v)$ is

$$\frac{\partial\mathcal{L}}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial v}\right) = 0\,. \tag{9}$$

Substituting equations (5)–(8) into equation (9), we find again equation (3). This demonstrates that action minimization is consistent with the force-balance calculation.

3 ASYMPTOTIC FORMS

In Section 2.1, we found the following asymptotic expressions for the acceleration a:

(a) *Newtonian limit* ($a \gg a_0$): $a \approx a_\mathrm{N} + a_0$. The constant a_0 amounts to a small correction to the Newtonian acceleration a_N (equation 4). Such a constant deviation from Newtonian dynamics has not been tested yet in the Cassini mission data (Hees et al. 2014, fitted only a quadrupolar correction to the Cassini data).

(b) *Deep MOND limit* ($a \ll a_0$): $a \approx \sqrt{a_\mathrm{N} a_0} + a_\mathrm{N}/2$. Using force balance $a = v^2/r$ and equations (4) and (7), this approximation takes the form

$$\frac{v^4}{M} \approx G_0 a_0\left[1 + \frac{r_M}{r} + \frac{1}{2}\left(\frac{r_M}{r}\right)^2\right]\,. \tag{10}$$

This equation represents the observed BTF/BFJ relations in which $v^4 = G_0 M a_0$ to zeroth order in $1/r$. It also shows that the quotient v^4/M is indeed related to MOND's universal constant $G_0 a_0$ which has precisely the same dimensions. The higher order correction terms decrease with distance r, thus they become negligible at large scales.

4 DISCUSSION

We have shown that a varying gravitational 'constant' $G(a) \propto 1/a$ in the radial Euler–Lagrange equation of the central force problem of Newtonian mechanics implies the observed BTF/BFJ relations at very low accelerations a. We have also derived a solution for the gravitational potential of a point mass (equation 5) and the acceleration of an orbiting test particle valid for all acceleration regimes (equation 3) using the same Lagrangian formulation (Section 2.2).

The adopted function for $G(a)$ has a unique form (equation 1) that describes correctly the behaviour of galactic stellar kinematics in the two asymptotic regimes (Newtonian and MOND). The same form also describes the intermediate regime in which we do not introduce any additional physics by avoiding the use of more complicated $G(a)$ functions with the same asymptotic behaviours. In addition, each of the two asymptotic terms in equation (1) generates a small contribution in the opposite limit: the Newtonian term $a_N/2$ modifies MOND's acceleration, whereas the term a_0 modifies the Newtonian acceleration (Section 3). Such a small non-Newtonian constant has been predicted in the weak-field limit of conformal Weyl gravity (Mannheim & Kazanas 1989, 1994).

It has been argued that a small quadrupolar correction to the Newtonian gravitational field of our Solar system, obtained from Cassini monitoring radio data, is consistent with relativistic deviations at large Solar system scales and offers no support for MOND-type deviations (Hees et al. 2014). In that investigation, a quadrupolar term was actually fitted to the Cassini data and its magnitude was estimated. These results are not applicable to our case, where the correction to the Newtonian acceleration within the Solar system is just a small constant term $a_0 \sim 1\,\text{Å}\,\text{s}^{-2}$ (Section 3). This correction is produced by a linear potential of the form $\delta\Phi(r) = +a_0\,r$. The Cassini data will have to be fitted again for this potential, although it may be difficult for the analysis to detect a correction as small as a_0 ($a_0/a_N \simeq 2 \times 10^{-6}$ at the distance of Saturn).

The above formulation of modified dynamics with $G(a)$ given by equation (1) shows that the only constant introduced in the deep MOND limit is G_0a_0. This unusual constant was already known to Milgrom (2015c) who argued that the product maintains scale invariance in MOND. But this is a mathematical argument and it implies that a_0 is not necessarily a constant of MOND in its deep limit. On the other hand, a_0 appears alone as a constant only in the Newtonian regime of accelerations, where $a \approx a_N + a_0$ (Section 3). In our modified dynamics, the new constant is introduced by the varying $G(a)$ itself. As such, the term G_0a_0/a does not have an obvious cosmological underpinning, it is rather localized to large scales in individual galaxies and it is in fact responsible for the appearance of the small Weyl-like correction a_0 to the acceleration in the Newtonian regime. Furthermore, it remains an open question whether G_0 and a_0 are separate constants among individual galaxies and clusters of galaxies. At such large scales, the individual values could possibly vary in a way that maintains a constant universal product G_0a_0.

Recently, Rodrigues et al. (2018) argued that a_0 cannot be constant in individual galaxies whose rotation curves were used to obtain its best-fitting value. On a statistical basis, a constant a_0 was rejected at more than the 10σ level of significance. This appears to be a much stronger result than from previous studies (Randriamampandry & Carignan 2014; Iocco, Pato & Bertone 2015; Hees et al. 2016) which also indicated that a_0 may not be constant between different galaxies. Rodrigues et al. (2018) concluded that MOND is not a viable theory on galactic scales. This conclusion is premature and it has already been disputed forcefully in the published literature (Li et al. 2018). If the above studies are confirmed by future independent investigations, the results may constitute evidence that G_0 and a_0 vary from galaxy to galaxy in a way that their product remains a universal constant. If true, such behaviour would make the study of gravitation in galaxies and galaxy clusters a lot more complicated.

ACKNOWLEDGEMENTS

We thank an anonymous referee for a thoughtful review of this paper. NASA support over the years is gratefully acknowledged.

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